

TOPOLOGICAL SYMMETRY GROUPS OF GRAPHS IN 3-MANIFOLDS

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ABSTRACT. We prove that for every closed, connected, orientable, irreducible 3-manifold, there exists an alternating group A_n which is not the topological symmetry group of any graph embedded in the manifold. We also show that for every finite group G , there is an embedding Γ of some graph in a hyperbolic rational homology 3-sphere such that the topological symmetry group of Γ is isomorphic to G .

1. INTRODUCTION

Characterizing the symmetries of a molecule is an important step in predicting its behavior. Chemists often model a molecule as a graph in \mathbb{R}^3 . They define the *point group* of a molecule as the group of isometries of \mathbb{R}^3 which take the molecular graph to itself. This is a useful way of representing the symmetries of a rigid molecule. However, molecules which are flexible or partially flexible may have symmetries which are not induced by isometries of \mathbb{R}^3 . Jon Simon [20] introduced the concept of the *topological symmetry group* in order to study the symmetries of such non-rigid molecules. This group has not only been used to study the symmetries of such molecules, but also the non-rigid symmetries of any graph embedded in S^3 . In this paper we extend this study to graphs embedded in other 3-manifolds.

In 1938, Frucht [6] showed that every finite group is the automorphism group of some connected graph. By contrast, in [5] we proved that the only finite simple groups which can occur as the topological symmetry group of a graph embedded in S^3 are cyclic groups and the alternating group A_5 . Thus, we can say that automorphism groups of connected graphs are *universal* for finite groups, while topological symmetry groups of graphs embedded in S^3 are not. We now prove that topological symmetry groups of graphs embedded in any given closed, connected, orientable, irreducible 3-manifold are not universal for finite groups.

In particular, let M be a 3-manifold, let γ be an abstract graph, and let Γ be an embedding of γ in M . Note that by a *graph* we shall mean a finite, connected graph with at most one edge between any pair of vertices and two

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distinct vertices for every edge. The *topological symmetry group* $\text{TSG}(\Gamma, M)$ is defined to be the subgroup of the automorphism group $\text{Aut}(\gamma)$ consisting of those automorphisms of γ which are induced by a homeomorphism of the pair (M, Γ) . Allowing only orientation preserving homeomorphisms defines the *orientation preserving topological symmetry group* $\text{TSG}_+(\Gamma, M)$. We prove the following result.

Theorem 1. *For every closed, connected, orientable, irreducible 3-manifold M , there exists an alternating group A_n which is not isomorphic to $\text{TSG}(\Gamma, M)$ for any graph Γ embedded in M .*

On the other hand, it follows from Frucht's theorem [6] that every finite group G can occur as the topological symmetry group of a graph embedded in some closed 3-manifold (which depends on G). We prove the following stronger result.

Theorem 2. *For every finite group G , there is an embedding Γ of a graph in a hyperbolic rational homology 3-sphere M such that $\text{TSG}(\Gamma, M) \cong G$.*

A graph is said to be *3-connected* if at least 3 vertices must be removed together with the edges they contain in order to disconnect the graph or reduce it to a single vertex. In [5], we showed that for every embedded 3-connected graph Γ embedded in S^3 , there is a subgroup of $\text{Diff}_+(S^3)$ isomorphic to $\text{TSG}_+(\Gamma, S^3)$. We now prove this is not the case for all 3-manifolds.

Theorem 3. *For every closed, orientable, irreducible, 3-manifold M that is not Seifert fibered, there is an embedding of a 3-connected graph Γ in M such that $\text{TSG}_+(\Gamma, M)$ is not isomorphic to any subgroup of $\text{Diff}_+(M)$.*

2. PROOF OF THEOREM 1

We begin by introducing some notation. Let Γ be an embedding of a graph in a 3-manifold M . Let V denote the set of embedded vertices of Γ , and let E denote the set of embedded edges of Γ . We construct a neighborhood $N(\Gamma)$ as the union of two sets, $N(V)$ and $N(E)$, which have disjoint interiors. In particular, for each vertex $v \in V$, let $N(v)$ denote a ball around v whose intersection with Γ is a star around v , and let $N(V)$ denote the union of all of these balls. For each embedded edge $e \in E$, let $N(e)$ denote a solid cylinder $D^2 \times I$ whose core is $e - N(V)$, such that $N(e) \cap \Gamma \subseteq e$, and $N(e)$ meets $N(V)$ in a pair of disks. Let $N(E)$ denote the union of all the solid cylinders $N(e)$. Let $N(\Gamma) = N(V) \cup N(E)$. We shall use $\partial'N(e)$ to denote the annulus $\partial N(\Gamma) \cap N(e)$ in order to distinguish it from the sphere $\partial N(e)$.

By the standard smoothing results in dimension 3 (proved in [14]), if a particular automorphism of an embedded graph Γ can be induced by an orientation preserving homeomorphism of (M, Γ) , then the same automorphism can be induced by an orientation preserving homeomorphism of (M, Γ) which is a diffeomorphism except possibly on the set of vertices of Γ . Thus we shall

abuse notation and call such a homeomorphism a *diffeomorphism* of (M, Γ) .

Lemma 1. *Let Γ be a graph embedded in ball $B \subseteq M$ an orientable, irreducible, 3-manifold, and suppose that $\text{TSG}_+(\Gamma, M)$ is a non-abelian simple group. Then $\text{TSG}_+(\Gamma, M) \cong A_5$.*

Proof. If $M = S^3$ the result follows from [5]. Thus we assume $M \neq S^3$. We obtain an embedding of Γ in S^3 by gluing a ball to the outside of B . Since Γ is connected, $S^3 - \Gamma$ is irreducible. Thus $M - \Gamma$ is the connected sum of the irreducible manifolds M and $S^3 - \Gamma$. Hence the splitting sphere ∂B is unique up to isotopy in $M - \Gamma$. Thus $\text{TSG}_+(\Gamma, M)$ is induced by a group G of orientation preserving diffeomorphisms of (M, Γ) which takes B to itself.

Since the restriction of G to B extends radially to S^3 taking Γ to itself, it follows that $\text{TSG}_+(\Gamma, M) \leq \text{TSG}_+(\Gamma, S^3)$. We show containment in the other direction as follows. Let p be a point of S^3 which is disjoint from Γ . Let g be an orientation preserving diffeomorphism of the pair (S^3, Γ) . We can compose g with an isotopy to obtain an orientation preserving diffeomorphism h of the pair (S^3, Γ) which pointwise fixes a neighborhood of p disjoint from Γ and induces the same automorphism of Γ as g . It follows that $\text{TSG}_+(\Gamma, S^3)$ can be induced by a group G' of orientation preserving diffeomorphisms of (S^3, Γ) which take the ball B to itself fixing its boundary pointwise. We may therefore restrict the elements of G' to B and then extend them to M by the identity. It follows that $\text{TSG}_+(\Gamma, M) = \text{TSG}_+(\Gamma, S^3)$, and hence by [5] we have $\text{TSG}_+(\Gamma, M) \cong A_5$. \square

Lemma 2. *Let Γ be a graph embedded in a closed, orientable, irreducible 3-manifold M such that $\text{TSG}_+(\Gamma, M)$ is a non-abelian simple group. Then there is a graph Λ embedded in M with $\text{TSG}_+(\Lambda, M) \cong \text{TSG}_+(\Gamma, M)$ such that $\partial N(\Lambda) - \partial' N(E)$ is incompressible in $\text{Cl}(M - N(\Lambda))$.*

Proof. The proof of this lemma is similar to the proof of a stronger result [5, Prop. 2] for $M = S^3$. We will therefore give only the main ideas of the argument, with many of the details omitted.

We say Γ has a *separating ball* B , if B meets Γ in a single vertex v with valence at least 3 and $\text{Int}(B)$ and $M - B$ each have non-empty intersection with Γ . In this case, we say $B \cap \Gamma$ is a *branch* of Γ at v .

Suppose that $\partial N(\Gamma) - \partial' N(E)$ is compressible in $\text{Cl}(M - N(\Gamma))$. Then there is a sphere in M meeting Γ in a single point such that each complementary component of the sphere intersects Γ non-trivially. Since M is irreducible, one of these components is ball. Also, Γ cannot be homeomorphic to an arc because $\text{TSG}_+(\Gamma, M)$ is non-abelian. Thus Γ has a separating ball at some vertex v .

First suppose, for the sake of contradiction, that $\text{TSG}_+(\Gamma, M)$ fixes v but does not setwise fix some branch Γ_1 at v . Let $\{\Gamma_1, \dots, \Gamma_n\}$ denote the orbit of Γ_1 under $\text{TSG}_+(\Gamma, M)$. Now the action of $\text{TSG}_+(\Gamma, M)$ on $\{\Gamma_1, \dots, \Gamma_n\}$ defines a non-trivial monomorphism $\Phi : \text{TSG}_+(\Gamma, M) \rightarrow S_n$. To see that Φ

is onto, let (ij) be a transposition in S_n . Since Γ_i and Γ_j are in the orbit of Γ_1 under $\text{TSG}_+(\Gamma, M)$, we can choose a pair of separating balls B_i and B_j which are disjoint except at v such that there is an orientation preserving diffeomorphism g of (M, Γ) with $g((B_i, \Gamma_i)) = (B_j, \Gamma_j)$. We can then choose a separating ball E containing $B_i \cup B_j$ such that $\partial E \cap (B_i \cup B_j) = \{v\}$. Now define an orientation preserving diffeomorphism $h : (E, \Gamma_i \cup \Gamma_j) \rightarrow E$ such that $h|_{B_i} = g|_{B_i}$, $h|_{B_j} = g^{-1}|_{B_j}$, and $h|_{\partial E}$ is the identity. Finally, extend h to $M - E$ by the identity. Thus we have $h : (M, \Gamma) \rightarrow (M, \Gamma)$ and h induces (ij) on Γ , and therefore Φ is onto. Since $\text{TSG}_+(\Gamma, M)$ is simple and non-abelian, this is impossible.

Let m denote an integer which is larger than the number of vertices in Γ . We will show that there is another graph Γ' in M such that $\text{TSG}_+(\Gamma', M) \cong \text{TSG}_+(\Gamma, M)$ and Γ' has fewer (possibly 0) branches at v . By repeating this argument as necessary one eventually obtains the embedded graph Λ . By the above paragraph, we only need to consider the following two cases.

Case 1: $\text{TSG}_+(\Gamma, M)$ fixes v and setwise fixes every branch of Γ at v .

Since $\text{TSG}_+(\Gamma, M)$ is non-trivial, there is some branch Γ_1 at v on which $\text{TSG}_+(\Gamma, M)$ acts non-trivially. Let e_1 denote an edge in Γ_1 containing v , and let $\{e_1, \dots, e_r\}$ be the orbit of e_1 under $\text{TSG}_+(\Gamma, M)$. We define Γ' as Γ_1 together with m vertices of valence 2 added to the interior of each e_i . We can define a non-trivial monomorphism $\psi : \text{TSG}_+(\Gamma, M) \rightarrow \text{TSG}_+(\Gamma', M)$ since each automorphism of Γ induces an automorphism of Γ' . Because of the m vertices on each e_i , every automorphism of Γ' fixes v . It is not hard to check that ψ is onto, and Γ' has fewer branches at v than Γ .

Case 2: v is not fixed by $\text{TSG}_+(\Gamma, M)$.

Let $\{\Gamma_1, \dots, \Gamma_n\}$ denote the orbit of Γ_1 under $\text{TSG}_+(\Gamma, M)$. Let e_1 denote an edge of the graph $\text{cl}(\Gamma - (\Gamma_1 \cup \dots \cup \Gamma_n))$ that contains v , and let $\{e_1, \dots, e_r\}$ and $\{v_1, \dots, v_q\}$ denote the orbits of e_1 and v respectively under $\text{TSG}_+(\Gamma, M)$. Now define Γ' as $\text{cl}(\Gamma - (\Gamma_1 \cup \dots \cup \Gamma_n))$ together with m vertices of valence 2 added to the interior of each e_i . We can define a non-trivial monomorphism $\psi : \text{TSG}_+(\Gamma, M) \rightarrow \text{TSG}_+(\Gamma', M)$ since each automorphism of Γ induces an automorphism of Γ' . Since, $\text{TSG}_+(\Gamma', M)$ cannot be cyclic, Γ' cannot be a simple closed curve. Thus every automorphism of Γ' leaves $\{v_1, \dots, v_q\}$ setwise invariant. Now it is not hard to check that ψ is onto and Γ' has fewer branches at v than Γ . \square

In addition to Lemmas 1 and 2, our proof of the next result will employ Jaco-Shalen [8] and Johansson's [9] theory of characteristic splittings along tori and annuli. For a survey of the definitions and statements see [1].

Proposition 1. *Let M be a closed, connected, orientable, irreducible 3-manifold. Then there exists an integer d depending only on M such that for any graph Γ embedded in M if $\text{TSG}_+(\Gamma, M)$ is a non-abelian simple group, then the order of $\text{TSG}_+(\Gamma, M)$ is at most d .*

Proof. We shall assume that Γ is not contained in a ball in M , since otherwise the result follows from Lemma 1. Since the proof is lengthy we divide it into six steps.

Step 1: We pick the number d .

Apply the Characteristic Submanifold Theorem [8, 9] to M to obtain a minimal collection Θ of incompressible tori in M such that the closure of each component of $M - \Theta$ is either atoroidal or Seifert fibered, and Θ is unique up to isotopy. If Θ is non-empty, let C_1, \dots, C_m denote the closures of the components of $M - \Theta$, otherwise let $m = 1$ and $C_1 = M$. Now we consider two cases.

Case 1: Either Θ is non-empty or Θ is empty and M is Seifert fibered.

Consider a component C_i which is Seifert fibered (possibly $C_i = M$). We see as follows that there is an upper bound n_i on the order of all finite simple non-abelian groups which can act faithfully on the base space of the fibration of C_i . If the base space is a sphere or projective plane (possibly with holes), then $n_i = 60$ is an upper bound since A_5 is the only finite simple non-abelian group which can act faithfully on a sphere or projective plane. If the base space has negative Euler characteristic χ , then $n_i = 84|\chi|$ is an upper bound by a classical result of Hurwitz [7]. Since no finite simple non-abelian groups can act faithfully on a torus or Klein bottle (possibly with holes), if the base space is one of these surfaces we let $n_i = 1$.

Now consider a component C_i which is not Seifert fibered. By the hypotheses of this case then $C_i \neq M$. It follows that C_i is atoroidal and has non-empty incompressible boundary. Hence, by Thurston's Hyperbolization Theorem [21], C_i admits a complete, hyperbolic structure with totally geodesic boundary. Now by Mostow's Rigidity Theorem [18], there is an integer n_i such that no finite group of diffeomorphisms of C_i has order greater than n_i .

After having chosen an n_i associated with each component C_i , we let $d = \text{Max}\{n_1, \dots, n_m, m!, 60\}$.

Case 2: Θ is empty and M is not Seifert fibered.

By the Geometrization Theorem [15, 16, 17], M has a geometric structure. Also since M is irreducible and not Seifert fibered, M does not admit a circle action. Furthermore, by the Elliptization Theorem [16], a 3-manifold with finite fundamental group is elliptic and hence Seifert fibered. Thus M has infinite π_1 , no circle action, and is irreducible. Hence by Kojima [11] there is a bound q on the order of finite groups of diffeomorphisms of M . In this case, we let $d = \text{Max}\{q, 60\}$.

Now let Γ be a graph embedded in M such that $\text{TSG}_+(\Gamma, M)$ is a simple non-abelian group. By Lemma 2, without loss of generality we can assume that $\partial N(\Gamma) - \partial' N(E)$ is incompressible in $\text{Cl}(M - N(\Gamma))$. We will prove that $\text{order}(\text{TSG}_+(\Gamma, M)) \leq d$.

Step 2: We choose $W \subseteq \text{Cl}(M - N(\Gamma))$ and a group G of diffeomorphisms of (M, Γ) leaving W setwise invariant.

Since Γ is connected and M is irreducible, the manifold $\text{Cl}(M - N(\Gamma))$ is irreducible. Thus we can apply the Characteristic Submanifold Theorem [8, 9] to the manifold $\text{Cl}(M - N(\Gamma))$ to get a minimal family of characteristic tori. When we split $\text{Cl}(M - N(\Gamma))$ along these tori, $\partial N(\Gamma)$ is contained entirely in one component, the closure of which we denote by X . Since $\partial N(\Gamma) - \partial' N(E)$ is incompressible in $\text{Cl}(M - N(\Gamma))$, Γ has no valence one vertices. Thus since $\text{TSG}_+(\Gamma, M)$ is not cyclic, it follows that Γ contains at least two simple closed curves. Therefore the genus of $\partial N(\Gamma)$ is at least two, and hence X cannot be Seifert fibered. It follows that X is atoroidal, and since M is irreducible, X is also irreducible.

Let P denote the set of annuli in $\partial' N(E)$ together with the torus boundary components of X . Since $\partial N(\Gamma) - \partial' N(E)$ is incompressible in $\text{Cl}(M - N(\Gamma))$, $\partial X - P$ is incompressible in X . Thus we can now apply the Characteristic Submanifold Theorem for Pared Manifolds [8, 9] to the pared manifold (X, P) . Since X is atoroidal, this gives us a minimal family Ω of incompressible annuli in X with boundaries in $\partial X - P$ such that if W is the closure of any component of $X - \Omega$, then the pared manifold $(W, W \cap (P \cup \Omega))$ is either simple, Seifert fibered, or I -fibered, and the set Ω is unique up to isotopy.

Let G denote the collection of orientation preserving diffeomorphisms of (M, Γ) which leave X , Ω , $N(V)$, and $N(E)$ setwise invariant. It follows from the uniqueness up to isotopy of each of these sets that every automorphism in $\text{TSG}_+(\Gamma, M)$ is induced by some element of G . Suppose that two elements of G induce the same automorphism on Γ . Then they induce the same permutation on the set of components of $\partial N(V)$. Since Γ has at most one edge between two vertices and every edge has two distinct vertices, they also induce the same permutation on the set of components of $\partial' N(E)$. It now follows that they induce the same permutation on the set of annuli in Ω as well as on the components of $X - \Omega$.

We construct a graph λ associated with (X, Ω) by representing the closure of each component of $X - \Omega$ by a vertex and defining an edge between a pair of vertices if the components they represent are adjacent. Observe that each annulus in Ω can be capped off by a pair of disks in $N(V)$ to obtain a sphere in M . Since M is irreducible, any sphere in M separates. It follows that all of the annuli in Ω separate X . Thus λ is a tree. Now G induces a group of automorphisms on λ which either fixes a vertex or fixes an edge setwise. Suppose that no vertex of λ is fixed by G . Then some edge of λ is inverted by an element of G . Hence there is an annulus $A \in \Omega$ which is setwise invariant under G and some element of G which interchanges the components adjacent to A . Thus we can define a non-trivial homomorphism $\varphi : \text{TSG}_+(\Gamma, M) \rightarrow \mathbb{Z}_2$ where $\varphi(a) = 1$ if and only if a is induced by an element of G which interchanges the components adjacent to A . However

this is impossible since $\text{TSG}_+(\Gamma, M)$ is simple and non-abelian. Hence there must be a vertex of λ which is fixed by G .

If the action of G on λ is non-trivial, then we choose a particular fixed vertex w of λ which is adjacent a vertex which is not fixed by G . In this case, let W be the closure of the component of $X - \Omega$ represented by w . Then W is setwise fixed by G and some annuli in $W \cap \Omega$ are not setwise fixed by G . Observe that all of the components of $W \cap \Omega$ are contained in a single component of ∂W . If this component of ∂W were a torus, then there would be a non-trivial homomorphism from $\text{TSG}_+(\Gamma, M)$ to \mathbb{Z}_r where r is the number of annuli in $W \cap \Omega$. Since $\text{TSG}_+(\Gamma, M)$ is a finite simple non-abelian group, this component of ∂W cannot be a torus.

If the action of G on λ is trivial, we choose W to be the closure of some component of $X - \Omega$ such that some components of $\partial N(V) \cap W$ are permuted by G . We know there is such a component of $X - \Omega$ since $\text{TSG}_+(\Gamma, M)$ is non-trivial and $\partial N(\Gamma)$ is contained in X . Observe that since the annuli of Ω separate X , and $\partial N(\Gamma)$ is connected, all of the components of $W \cap \partial N(V)$ are contained in a single component of ∂W . As above, if the component of ∂W containing $W \cap \partial N(V)$ were a torus, then there would be a non-trivial homomorphism from $\text{TSG}_+(\Gamma, M)$ to a finite cyclic group. Thus again, this component ∂W is not a torus.

Since not all of the components of ∂W are tori, W cannot be Seifert fibered. Thus the pared manifold $(W, W \cap (P \cup \Omega))$ is either I -fibered or simple.

Step 3: We prove that the pared manifold $(W, W \cap (P \cup \Omega))$ is simple.

Suppose, for the sake of contradiction, that $(W, W \cap (P \cup \Omega))$ is I -fibered. Then there is an I -bundle map of W over a surface such that $W \cap (P \cup \Omega)$ is the preimage of the boundary of the surface. It follows that the corresponding ∂I -bundle is $\partial N(V) \cap W$ and has either one or two components. If $\partial N(V) \cap W$ has two components which are interchanged by some element of G , then there would be a non-trivial homomorphism from $\text{TSG}_+(\Gamma, M)$ to \mathbb{Z}_2 . Thus we can assume that each component of $\partial N(V) \cap W$ is setwise invariant under G . It now follows from our definition of W that some annulus F_1 in $\Omega \cap W$ is not setwise invariant under G .

Let $\{F_1, \dots, F_r\}$ denote the orbit of F_1 under G . Since $r > 1$, the boundary components of F_1 do not co-bound an annulus in $\partial N(V) - W$. Thus we can cap off F_1 in $\partial N(V) - W$ to obtain a sphere. Let E_1 denote the closure of the component of the complement of this sphere in M which is disjoint from W . The orbit of E_1 under G is a pairwise disjoint collection E_1, \dots, E_r such that each $F_i \subseteq \partial E_i$. Suppose that E_1 is not a ball. Since M is irreducible, $\text{Cl}(M - E_1)$ is a ball containing E_2 . But since $E_2 \cong E_1$ is not a ball, this is impossible. Thus each E_i must be a ball.

Now the action of G on the orbit $\{F_1, \dots, F_r\}$ defines a non-trivial monomorphism $\Phi : \text{TSG}_+(\Gamma, M) \rightarrow S_r$. Furthermore, since $\text{TSG}_+(\Gamma, M)$ is non-abelian, $r > 2$. Hence the base surface of the I -bundle has at least three

boundary components. We see as follows that Φ is onto. Let (ij) be a transposition in S_r . Then there is a $g \in G$ such that $g(F_i) = F_j$. We saw above that each component of $\partial N(V) \cap W$ is setwise invariant under G . Hence a boundary component of F_i and its image under the element g are in the same component of $\partial N(V) \cap W$, and they project to distinct boundary components of the base surface of the I -bundle. Let N denote a regular neighborhood in the base surface of these two boundary components together with an arc between them. Then N is a disk with two holes. Now since M is orientable, the pre-image of N in the I -bundle is a product $N \times I$. We add the balls E_i and E_j to $N \times I$ along the annuli F_i and F_j respectively to obtain a solid cylinder $C \times I$.

We will define a homeomorphism $h : (M, \Gamma) \rightarrow (M, \Gamma)$ as follows. Let $h|_{E_i} = g|_{E_i}$ and $h|_{E_j} = g^{-1}|_{E_j}$. Then extend h within $N \times I$ so that h restricted to the cylinder $\partial C \times I$ is the identity and h leaves each of the disks $C \times \{0\}$ and $C \times \{1\}$ setwise invariant. Next we cap off the solid cylinder $C \times I$ in $N(V)$ to obtain a ball or pinched ball B whose boundary intersects Γ in either one or two vertices. Then extend h within B in such a way that $h|_{\partial B}$ is the identity and h leaves $\Gamma \cap B$ setwise invariant. Finally, we extend h to $M - B$ by the identity. Now by our construction, $h : (M, \Gamma) \rightarrow (M, \Gamma)$ is an orientation preserving homeomorphism such that $\Phi(h) = (ij)$. It follows that Φ is onto. However, this is impossible since $\text{TSG}_+(\Gamma, M)$ is a simple non-abelian group. Thus the pared manifold $(W, W \cap (P \cup \Omega))$ cannot be I -fibered, and hence must be simple.

Step 4: We define a group of isometries K of W and prove $K \cong \text{TSG}_+(\Gamma, M)$.

Since W is simple, it follows from Thurston's Hyperbolization Theorem for Pared Manifolds [21] applied to $(W, W \cap (P \cup \Omega))$ that $W - (W \cap (P \cup \Omega))$ admits a finite volume complete hyperbolic metric with totally geodesic boundary. Let D denote the double of $W - (W \cap (P \cup \Omega))$ along its boundary. Then D is a finite volume hyperbolic manifold, and every element of $\text{TSG}_+(\Gamma, M)$ is induced by an element of G whose restriction to W can be doubled to obtain a diffeomorphism of D . Now by Mostow's Rigidity Theorem [18], each such diffeomorphism of D is homotopic to an orientation preserving finite order isometry that restricts to an isometry of $W - (W \cap (P \cup \Omega))$. Furthermore, the set of all such isometries generates a finite group K of isometries of $W - (W \cap (P \cup \Omega))$. By removing horocyclic neighborhoods of the cusps of $W - (W \cap (P \cup \Omega))$, we obtain a copy of the pair $(W, W \cap (P \cup \Omega))$ which is contained in $W - (W \cap (P \cup \Omega))$ and is setwise invariant under the isometry group K . We shall abuse notation and consider K to be a finite group of isometries of $(W, W \cap (P \cup \Omega))$. Also K induces a finite group of isometries of the tori and annuli in $W \cap (P \cup \Omega)$ with respect to a flat metric. In particular, $\partial N(V) \cap W$, $\partial' N(E) \cap W$, and $\Omega \cap W$ are each setwise invariant under K . Finally, it follows from Waldhausen's

Isotopy Theorem [23] that each element of K is isotopic to an element of G restricted to W by an isotopy leaving $W \cap (P \cup \Omega)$ setwise invariant.

We show as follows that $\text{TSG}_+(\Gamma, M) \cong K$. Let $a \in \text{TSG}_+(\Gamma, M)$ be induced by the elements $g_1, g_2 \in G$. Then, as we observed in Step 2, g_1 and g_2 induce the same permutation of the components of $\partial N(V) \cap W$, $\partial' N(E) \cap W$, and $\Omega \cap W$. Now $g_1|_W$ and $g_2|_W$ are isotopic to some $f_1, f_2 \in K$ by isotopies leaving $W \cap (P \cup \Omega)$ setwise invariant. Thus f_1 and f_2 also induce the same permutation of the components of $\partial N(V) \cap W$, $\partial' N(E) \cap W$, and $\Omega \cap W$. Recall that the component of ∂W which meets $N(V)$ is not a torus. Thus there is some component J of $\partial N(V) \cap W$ which has $r \geq 3$ boundary components $\alpha_1, \dots, \alpha_r$. Now $f_1(J) = f_2(J)$ and $f_1(\alpha_i) = f_2(\alpha_i)$ for each $i = 1, \dots, r$. Hence $f_1^{-1} \circ f_2$ restricts to a finite order diffeomorphism of J which setwise fixes each component of ∂J . Since J is a sphere with at least three holes, this implies that $f_1^{-1} \circ f_2|_J$ is the identity. Finally, since f_1 and f_2 are isometries of W which are identical on the surface $J \subseteq \partial W$, it follows that $f_1 = f_2$. Thus the automorphism $a \in \text{TSG}_+(\Gamma, M)$ determines a unique element of K , and hence there is a well-defined homomorphism $\Phi : \text{TSG}_+(\Gamma, M) \rightarrow K$. Since every element of K came from such an element of $\text{TSG}_+(\Gamma, M)$, Φ is onto. Now since $\text{TSG}_+(\Gamma, M)$ is simple, it follows that $\text{TSG}_+(\Gamma, M) \cong K$.

Step 5: We extend K to a set W_2 whose boundaries are spheres that do not bound balls in $M - W_2$ and tori that are not compressible in $M - W_2$.

Every annulus in $W \cap (P \cup \Omega)$ separates X into two components. It follows that for any $\partial N(v)$ which meets W , each component of $\partial N(v) - W$ is either a disk or an annulus. Let V_1 denote the set of vertices of Γ such that each component of $N(V_1)$ meets W and let E_1 denote the set of edges such that each component of $N(E_1)$ meets W . We extend K to $\partial N(V_1) - W$ as follows. Extend each element of K radially within the disks components of $\partial N(V_1) - W$. Next consider an annulus component A of $\partial N(V_1) - W$. The boundaries of A must also be the boundary components of an annulus A' in $W \cap (P \cup \Omega)$. Since K restricts to a finite group of isometries of A' , we can extend K to a finite group of isometries of A . In this way we have extended K so that it is defined on each sphere in $\partial N(V_1)$. Now we extend K radially within each of the balls in $N(V_1)$ and in $N(E_1)$. Thus we have extended K to a finite group $K_1 \cong K$ acting faithfully on $W_1 = W \cup N(V_1) \cup N(E_1)$.

The boundary of W_1 consists of spheres and tori made up of the union of annuli in $\Omega \cap W$ with disks and annuli in $\partial N(V) - W$, together with the tori components of $\partial X \cap W$. Let $\{T_1, \dots, T_q\}$ denote the tori components of ∂W_1 which are compressible in $M - W_1$. Since the set $\{T_1, \dots, T_q\}$ is setwise invariant under G , the set $\{T_1, \dots, T_q\}$ must be setwise invariant under K_1 as well. Now we can choose a set of pairwise disjoint compressing disks $\{D_1, \dots, D_r\}$ for $\{T_1, \dots, T_q\}$ whose boundaries are setwise invariant under K_1 . Note that depending on the action of K on each T_i , we may have $r > q$. We add a product neighborhood of each D_i to W_1 to obtain a manifold whose

boundary contains more spheres than ∂W_1 but contains no tori which are compressible in $M - W_1$. We extend K_1 to these product neighborhoods by defining it radially within each parallel disk. Furthermore, for any boundary component of the union of W_1 together with these product neighborhoods which bounds a ball in $M - W_1$, we add that ball and extend K_1 radially within the ball. Thus we have extended K_1 to a finite group $K_2 \cong K_1$ acting faithfully on the manifold W_2 consisting of W_1 together with these product neighborhoods and balls. Observe that all of the components of ∂W_2 are either spheres which do not bound a ball in $M - W_2$ or tori which are not compressible in $M - W_2$.

Step 6: We prove $\text{order}(\text{TSG}_+(\Gamma, M)) = \text{order}(K) \leq d$ by considering 3 cases.

Case 1: Some component of ∂W_2 is a sphere S .

Since M is irreducible and S does not bound a ball in $M - W_2$, S must bound a ball B containing W_2 . Now any other sphere in ∂W_2 separates M such that the component of the complement which is disjoint from W_2 is contained in B . Since B is a ball, this component is also a ball. As this is contrary to our definition of W_2 , all of the components of ∂W_2 other than S must be tori. By gluing another ball to B we obtain S^3 such that each of the tori in ∂W_2 bounds a (possibly trivial) knot complement in S^3 that is disjoint from W_2 . We extend K_2 radially within the complementary ball. Then we replace each knot complement by a solid torus in such a way we obtain a homology sphere and we can extend K_2 radially within the solid tori. In this way we get an isomorphic finite simple non-abelian group K_3 of orientation preserving diffeomorphisms of a homology sphere. Now it follows from Zimmermann [24] that $K_3 \cong A_5$. Thus $\text{order}(K) = \text{order}(K_3) = 60 \leq d$.

Case 2: ∂W_2 has torus components but no spherical components.

Recall that every component of ∂W_2 is incompressible in $M - W_2$. Suppose that T is a torus component of ∂W_2 which is compressible in W_2 . Let N be a product neighborhood of a compressing disk in W_2 . By cutting T along $\partial N \cap T$ and capping off with the two disks in $\partial N - T$, we obtain a sphere S . Since T is incompressible in $M - W_2$, the component of $M - S$ which is disjoint from $W_2 - N$ is not a ball. Thus the component of $M - S$ containing $W_2 - N$ must be a ball. Hence the union of this ball and the neighborhood N is a solid torus V such that $\partial V = T$ and $W_2 \subseteq V \subseteq M$.

Let T_1, \dots, T_r denote the boundary components of W_2 , and suppose that every T_i is compressible in W_2 . Then, by the argument above, each T_i bounds a solid torus V_i such that $W_2 \subseteq V_i \subseteq M$. Now G induces an orientation preserving finite action on the solid tori V_1, \dots, V_r taking meridians to meridians, up to isotopy. Since G induces a finite action on the set of tori $\{T_1, \dots, T_r\}$ on the level of homology, this means there is also a set of longitudes $\{\ell_1, \dots, \ell_r\}$ which are setwise invariant under G up to isotopy. Hence the action that K_2 induces on the set of tori $\{T_1, \dots, T_r\}$ leaves the set of longitudes $\{\ell_1, \dots, \ell_r\}$ setwise invariant up to isotopy.

We obtain a homology sphere $W_3 = W_2 \cup U_1 \cup \dots \cup U_r$ by gluing a solid torus U_i along each boundary component T_i of W_2 so that a meridian μ_i of U_i is glued to the longitude ℓ_i . Thus K_2 leaves the set of meridians $\{\mu_1, \dots, \mu_r\}$ setwise invariant up to isotopy. Now since the action of K_2 on the set of tori $\{\partial U_1, \dots, \partial U_r\}$ is finite, for some $q \geq r$, we can find a set of pairwise disjoint meridians $\{m_1, \dots, m_q\}$ for the solid tori $\{U_1, \dots, U_r\}$ which is setwise invariant under K_2 . Now extend K_2 radially from the set of meridians $\{m_1, \dots, m_q\}$ to a set of pairwise disjoint meridional disks for the solid tori U_1, \dots, U_r . These meridional disks cut the solid tori U_1, \dots, U_r into a set of solid cylinders, and hence we can also extend K_2 radially within this set of solid cylinders. In this way we obtain a finite group $K_3 \cong K_2$ acting faithfully on the homology sphere W_3 . Thus it again follows from Zimmerman [24] that $K_3 \cong A_5$, and hence $\text{order}(K) = \text{order}(K_3) = 60 \leq d$.

Therefore, we can assume that some component of ∂W_2 is an incompressible torus in M . If a torus component of ∂W is setwise fixed by K_2 , then K_2 restricts to a faithful action of the torus, which is impossible since K_2 is a finite simple non-abelian group. Thus some incompressible boundary component of ∂W_2 has non-trivial orbit $\{T_1, \dots, T_q\}$ under K_2 . Now either each T_i is isotopic to a torus in the characteristic family Θ or each T_i is vertical in a closed up Seifert fibered component of $M - \Theta$. Suppose that each T_i is isotopic to a torus in Θ . Then without loss of generality we can assume that each T_i is in Θ . It follows that there is a non-trivial monomorphism from K_2 to S_m (where m is the number of tori in Θ). Hence $\text{order}(K) = \text{order}(K_2) \leq m! \leq d$.

Thus we can assume that each T_i is a vertical torus in a closed up Seifert fibered component of $M - \Theta$ and T_i is not isotopic to a torus in Θ . Let C_1, \dots, C_r denote all of the closed up Seifert fibered components of $M - \Theta$. The action of K_2 on the set $\{T_1, \dots, T_q\}$ induces an action on the set $\{C_1 \cap W_2, \dots, C_r \cap W_2\}$, which in turn defines a homomorphism from K_2 to S_r . Since K_2 is simple, either this homomorphism is trivial or $\text{order}(K) = \text{order}(K_2) \leq r! \leq m! \leq d$.

Therefore, we can assume that the homomorphism is trivial, and hence the orbit $\{T_1, \dots, T_q\}$ is contained in a single Seifert fibered component C . Now there is a non-trivial monomorphism from K_2 to S_q . Since K_2 is simple and non-abelian, $q \geq 5$. In particular, the Seifert fibered space $C \cap W_2$ is a Haken manifold with more than two boundary components. Now it follows from Waldhausen [22] that the fibration on $C \cap W_2$ is unique up to isotopy. Hence by Meeks and Scott [13], $C \cap W_2$ has a K_2 -invariant fibration. Thus K_2 induces an action on the base surface of the fibration of $C \cap W_2$. Let F be the base surface for the Seifert fibration of C , and let F' be the base surface for the Seifert fibration of $C \cap W_2$. Since the vertical tori T_i are incompressible in C , none of the boundary components of F' bounds a disk in F . Furthermore, since there are at least five such tori, $\chi(F) \leq \chi(F') < 0$. Note that since the action of K_2 on the set $\{T_1, \dots, T_q\}$ is non-trivial, K_2 cannot take each fiber of $C \cap W_2$ to itself.

Thus K_2 induces an isomorphic action on F' . Now using Hurwitz [7], we have $\text{order}(K) = \text{order}(K_2) \leq 84|\chi(F')| \leq 84|\chi(F)| \leq d$.

Case 3: ∂W_2 is empty.

In this case, W_2 is the 3-manifold M . Suppose that the set of characteristic tori Θ is non-empty. Since Θ is unique up to isotopy, we can find an isotopic set of tori Θ' which are setwise invariant under K_2 . Thus there is a homomorphism from K_2 to S_m (recall that m is the number of tori in Θ). Furthermore, since K_2 is a non-trivial finite simple non-abelian group of orientation preserving diffeomorphisms of W_2 , K_2 cannot leave a torus setwise invariant. Thus the homomorphism is injective, and hence $\text{order}(K) = \text{order}(K_2) \leq m! \leq d$.

Therefore we can assume that Θ is empty. If M is not Seifert fibered, then $\text{order}(K) = \text{order}(K_2) \leq d$ by Case 2 of Step 1. So we can further assume that M is Seifert fibered. Now by Waldhausen [22], if M is a closed Haken manifold other than the 3-torus and the double of the twisted I -bundle over a Klein bottle, then M has a fibration which is unique up to isotopy. Also, by Ohshika [19], if M is a non-Haken manifold with infinite π_1 , then M has a fibration which is unique up to isotopy. Since M is irreducible, if the fibration is unique up to isotopy, then we can apply Meeks and Scott [13] to get a K_2 -invariant fibration. In this case, we can argue as in the end of Case 2 to again conclude that $\text{order}(K) \leq d$.

Thus we can assume that either M is the 3-torus, M is the double of the twisted I -bundle over a Klein bottle, or M has finite fundamental group. Observe that the 3-torus and the twisted I -bundle over a Klein bottle both have flat geometry. By Meeks and Scott [13] any smooth finite group action of a flat manifold preserves the geometric structure. However, by considering the lift of the action to \mathbb{R}^3 we see that no finite simple non-abelian group can act geometrically and faithfully on either the 3-torus or the twisted I -bundle over a Klein bottle. Thus M cannot be either of these manifolds.

Finally, suppose that M has finite fundamental group. Now by the proof of the Elliptization Conjecture [16], M has elliptical geometry, and hence by Dinkelbach and Leeb [3] we can assume that K_2 acts geometrically on M . However by the classification of orientation preserving isometry groups of elliptic 3-manifolds of Kalliongis and Miller [10] and McCullough [12], no finite simple non-abelian geometric group action of an elliptic 3-manifold has order greater than 60. Thus $\text{order}(K_2) \leq 60 \leq d$. \square

Proof of Theorem 1. Let d be the number given by Proposition 1 for the manifold M and choose $n > d$. Then the alternating group A_n is a non-abelian simple group, and by Proposition 1 no embedding of any graph Γ in M has $\text{TSG}_+(\Gamma, M) \cong A_n$. Now since $\text{TSG}_+(\Gamma, M)$ is either equal to $\text{TSG}(\Gamma, M)$ or is a normal subgroup of $\text{TSG}(\Gamma, M)$ of index 2, there is no embedding of a graph Γ in M such that $\text{TSG}(\Gamma, M) \cong A_n$. \square

3. PROOFS OF THEOREMS 2 AND 3

In contrast with Theorem 1, we prove in Theorem 2 that if the manifold M can vary (even among the hyperbolic rational homology spheres) then the collection of topological symmetry groups of embedded graphs is universal. We begin with the following proposition.

Proposition 2. *Let M be a connected 3-manifold, and let G be a finite group of diffeomorphisms acting freely on M . Then there is a graph Λ graph embedded in M such that $\text{TSG}(\Lambda, M) \cong G$.*

Proof. Let $n = \text{order}(G)$. If $n = 1$ or 2 , we can choose Λ to be a single vertex or a single edge in M , respectively. Thus we assume that $n > 2$.

Let U , V , and W be sets of n vertices each. Let γ be the abstract graph with vertices in $U \cup V \cup W$ and an edge between a pair of vertices if and only if precisely one of the vertices is in V . Then every automorphism of γ leaves the set V setwise invariant since the valence of the vertices in V is twice that of the vertices in $U \cup W$. It follows that if an automorphism of γ setwise fixes an edge, then it fixes both vertices of that edge.

We embed γ in M as follows. Let u , v , and w be points in M whose orbits under G are disjoint. Embed the sets U , V , and W as the orbits of u , v , and w respectively under G . We abuse notation and refer to both the abstract and embedded sets of vertices as U , V , and W . Since G acts freely on M , G induces a faithful action of the abstract graph γ such that no non-trivial element of G fixes any vertex or inverts any edge of γ . Furthermore, the quotient map $\pi : M \rightarrow M/G$ is a covering map and M/G is a 3-manifold.

Let $\{\varepsilon_1, \dots, \varepsilon_m\}$ consist of one representative from each orbit of the edges of the abstract graph γ under G , and for each i let x_i and y_i denote the embedded vertices of ε_i . Since M is path connected, for each i we can choose a path α_i in M from x_i to y_i and let $\alpha'_i = \pi(\alpha_i)$. Since G leaves each of U , V , and W setwise invariant, each α'_i has distinct endpoints. Now, by general position in M/G , we can homotop each α'_i fixing its endpoints to a simple path ρ'_i such that the interiors of the ρ'_i are pairwise disjoint and are disjoint from $\pi(V \cup U \cup W)$. For each i , let ρ_i denote the lift of the path ρ'_i beginning at x_i . Then ρ_i is a simple path in M , and since ρ'_i is homotopic fixing its endpoints to α'_i , the other endpoint of ρ_i is y_i . For each i , embed the abstract edge ε_i as the image of ρ_i in M .

Now let ε be an arbitrary edge of γ . Since no edge of γ is setwise fixed by a non-trivial element of G , there is a unique $g \in G$ and i such that $\varepsilon = g(\varepsilon_i)$. Hence we can unambiguously embed ε as $g(\rho_i)$. Let Γ consist of the embedded vertices $V \cup U \cup W$ together with embeddings of the edges of γ defined in this way. It follows from our choice of the paths ρ'_i in M/G that these embedded edges are pairwise disjoint and their interiors are disjoint from the set of vertices $V \cup U \cup W$. Thus Γ is indeed an embedding of γ in M , and is setwise invariant under G .

Now let the set $\{\rho_1, \dots, \rho_m\}$ consist of one representative from each orbit of the embedded edges of Γ under G . We create a new embedded graph Λ

by adding i vertices of valence 2 to the interior of every edge in the orbit of ρ_i in such a way that G leaves Λ setwise invariant. Then G induces a faithful action on Λ , and hence is isomorphic to a subgroup of $\text{TSG}(\Lambda, M)$.

We prove as follows that $G \cong \text{TSG}(\Lambda, M)$. Let h be a homeomorphism of M inducing a non-trivial automorphism of Λ . Since Γ has no vertices of valence 2, h leaves Γ setwise invariant inducing a non-trivial automorphism of Γ . Hence there is some edge e of Γ such that $h(e) \neq e$. Now e is in the orbit of some ρ_i under G , and hence as a path in Λ , e contains precisely i vertices of valence 2. Thus $h(e)$ also contains precisely i vertices of valence 2, and hence is also in the orbit of ρ_i under G . It follows that for some $g \in G$, $g(e) = h(e)$. Now $f = g^{-1}h$ is a homeomorphism of (M, Γ) taking e to itself, and hence fixing both vertices of e as an edge in Γ .

Suppose, for the sake of contradiction, that there is some edge e' adjacent to e such that $f(e') \neq e'$. Since f fixes both vertices of e , f must fix the vertex $x = e \cap e'$. By repeating the above argument with $f(e')$ instead of $h(e)$, we see that there is a $g_1 \in G$ such that $g_1(e') = f(e')$. However, since G acts freely on M , g_1 cannot fix x . Thus $g_1(e')$ has vertices x and $g_1(x)$. Since $g_1(e')$ is an edge of Γ , precisely one of its vertices is contained in V . But this is impossible since g_1 leaves V setwise invariant. Thus $f(e') = e'$, and hence inductively we see that f leaves every edge of Γ setwise invariant. Since f cannot interchange the vertices of any edge of Γ , f induces the trivial automorphism on Γ and hence on Λ as well. Thus, g induces the same automorphism as h on Λ . It follows that $\text{TSG}(\Lambda, M) \cong G$. \square

Now Theorem 2 follows immediately from Proposition 2, since Cooper and Long [2] have shown that for every finite group H , there is a hyperbolic rational homology 3-sphere M with a group of diffeomorphisms $G \cong H$ such that G acts freely on M .

It was proved in [5] that if a 3-connected graph Γ is embedded in S^3 , then $\text{TSG}_+(\Gamma, S^3)$ is isomorphic to a subgroup of the group of orientation preserving diffeomorphisms $\text{Diff}_+(S^3)$. We show below that this is not true for all 3-manifolds.

Proof of Theorem 3. By the Geometrization Theorem [15, 16, 17], M can be decomposed into geometric pieces. Also, since M is irreducible and not Seifert fibered, M does not admit a circle action. Furthermore, by the Elliptization Theorem [16], a 3-manifold with finite fundamental group is elliptic and hence Seifert fibered. Thus M has infinite π_1 , has no circle action, and is irreducible. Hence by Kojima [11] there is a bound on the order of finite groups of diffeomorphisms of M . In particular, there is a prime $p > 3$ such that \mathbb{Z}_p is not a subgroup of $\text{Diff}_+(M)$. Now it follows from [4] that there is an embedding Δ of the complete graph K_p in the interior of a ball B such that (B, Δ) has an orientation preserving diffeomorphism h which induces an automorphism of order p on Δ . Since h is orientation preserving h is isotopic to the identity on B . Thus we can modify h by an isotopy to get a diffeomorphism g of (B, Δ) such that $g|_{\partial B}$ is the identity

and g induces an automorphism of order p on Δ . Now we embed B in M , and extend g to M by the identity. This gives us an embedding Γ of K_p in M with $\mathbb{Z}_p \leq \text{TSG}_+(\Gamma)$. It follows that $\text{TSG}_+(\Gamma)$ cannot be isomorphic to any finite subgroup of $\text{Diff}_+(M)$. Finally, since $p > 3$, K_p is 3-connected. \square

REFERENCES

- [1] F. Bonahon, *Geometric structures on 3-manifolds*, Handbook of Geometric Topology, 93–164, North-Holland, Amsterdam, 2002.
- [2] D. Cooper and D. Long, *Free actions of finite groups on rational homology 3-spheres*, Topology Appl. **101** (2000), 143–148.
- [3] J. Dinkelbach and B. Leeb, *Equivariant Ricci flow with surgery and applications to finite group actions on geometric 3-manifolds*, Geom. Topol. **13** (2009), 1129–1173.
- [4] E. Flapan, *Rigidity of graph symmetries in the 3-sphere*, J. Knot Theory Ramifications **4** (1995), 373–388.
- [5] E. Flapan, R. Naimi, J. Pommersheim, and H. Tamvakis, *Topological symmetry groups of embedded graphs in the 3-sphere*, Comment. Math. Helv. **80** (2005), 317–354.
- [6] R. Frucht, *Herstellung von Graphen mit vorgegebener abstrakter Gruppe*, Compositio Math. **6** (1939), 239–250.
- [7] A. Hurwitz, *Über algebraische Gebilde mit Eindeutigen Transformationen in sich*, Math. Annalen **41** (1893), 403–442.
- [8] W. H. Jaco and P. B. Shalen, *Seifert Fibred Spaces in 3-manifolds*, Mem. Amer. Math. Soc. **21** (1979), no. 220.
- [9] K. Johannson, *Homotopy equivalences of 3-manifolds with boundaries*, Lecture Notes in Mathematics, **761**. Springer, Berlin, (1979).
- [10] J. Kalliongis and A. Miller, *Geometric group actions on lens spaces*, Kyungpook Math. J. **42** (2002), 313–344.
- [11] S. Kojima, *Bounding finite groups acting on 3-manifolds*, Math. Proc. Cambridge Philos. Soc. **96** (1984), 269–281.
- [12] D. McCullough, *Isometries of elliptic 3-manifolds*, J. London Math Soc. **65** (2002), 167–182.
- [13] W. Meeks and P. Scott, *Finite group actions on 3-manifolds*, Invent. Math. **86** (1986), 287–346.
- [14] E. E. Moise, *Geometric Topology in Dimensions 2 and 3*, Springer-Verlag New York (1977).
- [15] J. W. Morgan and F. T.-H. Fong, *Ricci Flow and Geometrization of 3-manifolds*, University Lecture Series **53**, American Mathematical Society, Providence, RI, (2010).
- [16] J. Morgan and G. Tian, *Ricci flow and the Poincare conjecture*, Clay Mathematics Monographs, **3**, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007.
- [17] J. Morgan and G. Tian, *Completion of the proof of the Geometrization Conjecture*, arXiv:0809.4040
- [18] G. Mostow, *Strong Rigidity of Locally Symmetric Spaces*, Annals of Mathematics Studies **78**, Princeton University Press, Princeton, NJ, 1973.
- [19] K. Ohshika, *Teichmüller spaces of Seifert fibered Manifolds with infinite π_1* , Topology Appl. **27** (1987), 75–93.
- [20] J. Simon, *Topological chirality of certain molecules*, Topology **25** (1986), 229–235.
- [21] W. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982), 357–381.
- [22] F. Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten II*, Invent. Math **4** (1967), 87–117.

- [23] F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. Math. **87** (1968), 56–88.
- [24] B. Zimmerman, *On finite simple groups acting on homology 3-spheres*, Topology Appl. **125** (2002), 199–202.

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